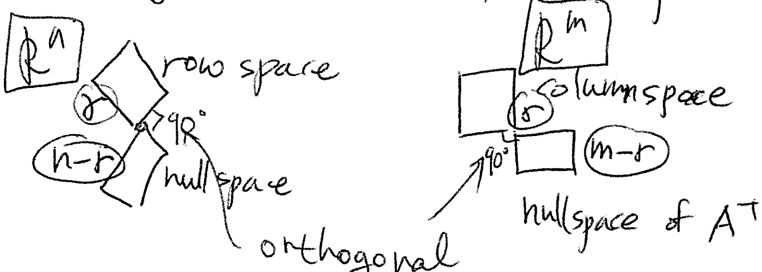
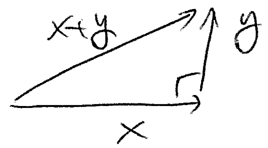
• Orthogonal Vectors of Subspaces.





Pythagoras.

→ $x^T y = 0$

↳ $\|x\|^2 + \|y\|^2 = \|x+y\|^2$ (14, 5, 19)

$x^T x + y^T y = (x+y)^T (x+y)$

$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

↳ → $x^T y = 0$

"subspace S is orthogonal to subspace T"

means... "every vector in S is orthogonal to every vector in T"

★ row space is orthogonal to null space of A

why? $Ax = 0$ $\begin{bmatrix} \text{row}_1 \text{ of } A \\ \text{row of } A \\ \vdots \\ \text{row}_n \text{ of } A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

⇓ x is orthogonal to all the rows. & all their combinations

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$n = 3$ $r = 1$ $\dim N(A) = 2$.

"null space and row space are orthogonal complements in R^n"

⇔ "Null contains all vectors ⊥ row space"

Coming ; $Ax = b$

"Solve" when there is NO solution.

(ⓐ $m > n$ (rank < m)

Then, what is the BEST solution?

★ $A^T A$?

① n×m m×n → n×n square.

② symmetric $(A^T A)^T = A^T A$

③ Is it invertible?

$Ax = b$ ↓ best solution.

$A^T A \hat{x} = A^T b$

ex

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ → most $\vec{b}$ (column vectors are not in the plane...) → no solution..

⇓

r=2 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$

⇑ invertible!

r=1 $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$

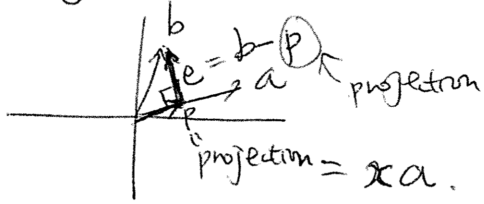
⇑ not invertible!

$N(A^TA) = N(A)$

rank of A^TA = rank of A.

° "A^TA is invertible exactly if A has indep. columns"

★ Projection!



$a^T(b - xa) = 0.$

$x a^Ta = a^Tb$

$\rightarrow x = \frac{a^Tb}{a^Ta}$, $p = ax = a\frac{a^Tb}{a^Ta} = \frac{aa^T}{a^Ta} b$ (n×n matrix)

If... $b \rightarrow 2b$: $p \rightarrow 2p$

$a \rightarrow 2a$; $p \rightarrow p$

$p = Pb$ ← projection matrix 라고 해보자!

★ $P = \frac{aa^T}{a^Ta}$

property 1: $C(P)$ = line through a. (column space of P)

∴ rank$(P) = 1$.

property 2: $P^T = P$: symmetric.

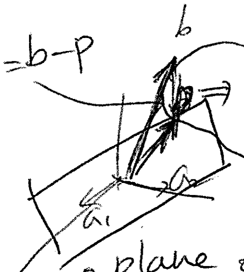
property 3: $P(p) = P(Pb) = P^2b$ $P^2 = P$

Q. Why Project?

Because $Ax = b$ may have no solution!

↓ solve $A\hat{x} = p$ instead! (p → projection of b onto columnspace.)

$e = b - p$



the nearest point of b in the columnspace

° plane of a_1, a_2 = col space of $A = [a_1\ a_2]$

° $e = b - p$ is perpendicular to the plane. (error vector)

° $p = x_1a_1 + x_2a_2$

$p = A\hat{x}$

key : $b - A\hat{x}$ is perp. to plane. (= e = error vector)

$\Rightarrow a_1^T(b - A\hat{x}) = 0$, $a_2^T(b - A\hat{x}) = 0$

$\Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow A^T(b - A\hat{x}) = 0.$ ← e

∴ e is in $N(A^T)$

⟺ $e \;\varepsilon\; C(A)$ (columnspace of A) Yes!!

$\Rightarrow A^TA\hat{x} = A^Tb$ (A^TA: n×n)

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x} = A(A^TA)^{-1}A^Tb.$

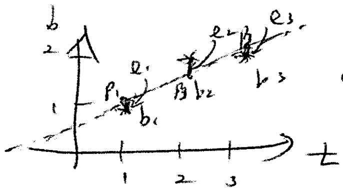
$\left[1D \rightarrow \frac{aa^T}{a^Ta} \right]$

projection

① matrix $P = A(A^TA)^{-1}A^T$

↳ $P^T = P$ & $P^2 = P$

* Least squares fitting by line.



$b = C + Dt$

(3-points) (1,1) (2,2) (3,2)

↳ (3 equations)

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2.$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

How to understand ?? ($P = A(A^TA)^{-1}A^T$)

① If b in column space $Pb = b$

If b in ⊥ column space $Pb = 0$. (Why?)

↳ null space of A^T

① If $Pb = b$ ($b \in C(A)$)

→ $Pb = [A(A^TA)^{-1}A^T]\, b = [A(A^TA)^{-1}A^T][Ax]$

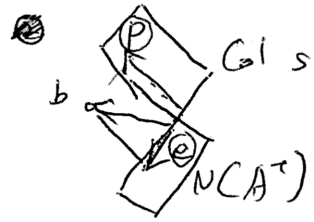
(b는 A column vector combination. x는 coefficient vector)

$= A(A^TA)^{-1}A^TA\, x$

$= Ax$

$= b.$ ∴ $Pb = b$

②



$b = p + e$

$\Rightarrow e = (I - P)\, b$

↑ proj onto ⊥ space

minimize $\|Ax - b\|^2 = \|e\|^2$

$= e_1^2 + e_2^2 + e_3^2$

$= (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

(= linear regression)

Find $x = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, p

$A^TA\hat{x} = A^Tb$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

다시정리 ⇓ b가 $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ 계산하기.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

$\begin{cases} 3C + 6D = 5 \\ 6C + 14D = 11 \end{cases}$

↳ Normal equation

$2D = 1$

↳ $C = \frac{2}{3}$, $D = \frac{1}{2}$

$b = p + e$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

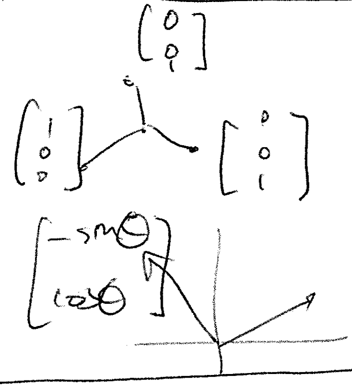
(prove) If A has independent columns then A^TA is invertible.

Suppose $A^TA x = 0$ ⟶ x must be zero.

(x^T 곱해.)

Trick: $x^TA^TAx = 0$
(IDEA) ↳ $(Ax)^T(Ax) \Rightarrow Ax = 0 \Rightarrow x = 0.$
↑ (A's column ind.)

"Columns definitely independent if they are perpendicular unit vectors"



orthonormal.

Orthonormal vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = I$$

If Q is square then $Q^TQ = I$,
↗ tells us $Q^T = Q^{-1}$

Orthogonal matrix (orthonormal matrix??)

- permutation $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
  ↳ $Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow Q^TQ = I.$

(ex1) $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(ex2) $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(ex3) $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$
↳ Adhemar matrix

(ex) Rectangular
↳ $Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$ → Rectangular to square by using Gram-Schmit.
→ $\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

Suppose Q has orthonormal columns
→ project onto its column space
→ $P = Q(Q^TQ)^{-1}Q^T = QQ^T \stackrel{?}{=} I$ (if Q is square)
(Q^TQ = I (orthogonal))

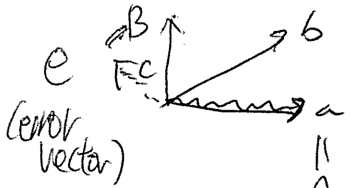
$(QQ^T)(QQ^T) = QQ^T$

normal equation $A^TA\hat{x} = A^Tb$
↓
Now A is Q: $Q^TQ\hat{x} = Q^Tb$ ($Q^TQ = I$)

$$\hat{x}_i = q_i^Tb$$

*Gram-Schmidt

· independent vectors (a, b, c) → orthogonal



A, B, C
↓
orthonormal

$q_1 = \frac{A}{\|A\|},\ q_2 = \frac{B}{\|B\|},\ q_3 = \frac{C}{\|C\|}$

$A = a$

$B = b - \frac{A^T b}{A^T A} A$

$A^T B = A^T\left(b - \frac{A^T b}{A^T A} A\right) = 0$

$C = c - \frac{A^T c}{A^T A} A\,(?) - \frac{B^T c}{B^T B} B\,(?)$

↑ $C \perp A$ ↑ $C \perp B$

ex)

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $c =$

$a \parallel A$

$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$Q = [q_1\ q_2] = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

$A = LU$ 기억...

$A = QR$

$[a_1\ a_2] = [q_1\ q_2]\begin{bmatrix} a_1^T q_1 & * \\ a_1^T q_2 & * \end{bmatrix}$, $a_1^T q_2 = 0$

*Determinant.

property

① $\det I = 1$.

② Exchange rows: reverse the sign of the DET.

$\det(P) = 1$ or -1 (even) (odd)

↑ permutation

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

③a $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

③b $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

∴ det. is linear fcn to row vector.

(! $\det(A+B) \neq \det A + \det B$)

② → ④ 2 equal rows → det = 0

∵ Exchange those rows → same matrix

⑤ Subtract $l \times$ row i from row k

→ DET does not change.

$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \overset{③}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$

↑ ④ zero

(6) Row of zeros → $\det A = 0$.

(7) $U = \begin{bmatrix} d_1 & \times & \times & \times \\ & \ddots & & \times \\ 0 & & \ddots & \times \\ & & & d_n \end{bmatrix}$ (triangular upper)

$\det U = \prod_{i}^{n} d_i$ (product of pivots)

(8) $\det A = 0$ exactly when A is singular
(non-invertibility)

⇒ If $\det A = 0$ → invertible.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \to \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} \to \begin{vmatrix} a & 0 \\ 0 & d - \frac{c}{a}b \end{vmatrix} = ad - bc$

(9) $\det AB = \det(A)\det(B)$

$\det A^{-1} = 1/\det(A)$

$\det 2A = 2^n \det A$

(10) $\det A^T \overset{?}{=} \det A$

$|U^T L^T| \overset{?}{=} |LU|$

$|U^T||L^T| \overset{?}{=} |L||U|$
diag. diag. diag. diag.

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} \cdots$

$= a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} \cdots$

BIG FORMULA.

$\det A = \sum_{n!\ \text{terms}} (\pm)\, a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$

$(\alpha, \beta, \gamma, \cdots \omega)$ = permutation of $(1, 2 \cdots n)$

ex $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0$.

$(4, 3, 2, 1)$ $(3, 2, 1, 4)$
↓ ↓
1 −1

Cofactors 3×3. IN PARENS

$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$
$+ a_{12}(\cdots)$
$+ a_{13}(\cdots)$

$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

- Cofactor of $a_{ij} = C_{ij}$

$\pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \end{pmatrix}$

(+) $i+j$ even (−) $i+j$ odd.

Cofactor formula (along row 1)

$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{1n}C_{1n}$

ex $A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ $|A_1| = 1$ $|A_2| = 0$ $|A_3| = -1$

$|A_4| = 1|A_3| - 1(A_2) = -1$

$|A_5| = 0$

$|A_6| = 1$

$|A_7| = 1$

Inverse matrix ~~DET~~

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T$$

C: cofactor matrix ← products of $n-1$ entries.

$\det A$ ← products of n entries

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}$$

- Check $AC^T = (\det A) I$.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \hline \\ \hline a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & & C_{n1} \\ C_{12} & & \vdots \\ \vdots & & \vdots \\ C_{1n} & & C_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} \det(A) & & 0 \\ & \det(A) & \\ 0 & & \ddots \quad \det(A) \end{bmatrix}$$

off-diagonal = 0

i.e.) $i \neq j \rightarrow$ (ith row of A) × (jth col of C) = 0.

why?

$$Ax = b$$

$$x = A^{-1}b = \frac{1}{\det A} C^T b$$

good tip!

$$C_{11}b_1 + C_{21}b_2 + \cdots$$

- Cramer's Rule

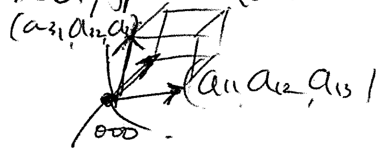
$$x_1 = \frac{\det B_1}{\det A}$$

$$x_2 = \frac{\det B_2}{\det A}$$

A with column 1 replaced by b → $B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \text{ columns} \\ \text{of } A \end{matrix} \end{bmatrix}$

B_j = A with column j replaced by b

(3x3) $|\det A|$ = volume of box.

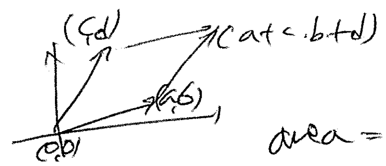


A = I → ok → generalize.

A = Q → orthogonal matrix.

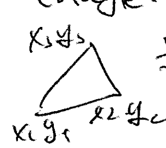
$|\det A|$ = volume of box. 1, 2, 3a, 3b (?)

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



$$\text{area} = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

triangle.



$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

Kangee

{Eigenvalues & Eigenvectors}

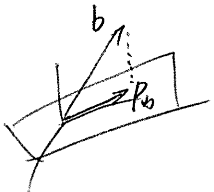
$\det[A - I\lambda] = 0$.

Ax parallel to x (Eigenvectors)

$\therefore$ $Ax = \lambda x$

↓ eigenvalue ↘ eigenvectors.

If A is singular, $\lambda = 0$ is eigenvalue



What are x's and λ's for projection matrix?

→ Any x in plane : $Px = x$ → $\lambda = 1$

→ Any $x \perp$ plane : $Px = 0$ → $\lambda = 0$

peramutation matrix?

→ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 1$.

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda = -1$.

- Fact : sum of λ's $= a_1 + a_2 + \cdots a_n$

Q. How to solve $Ax = \lambda x$.

Rewrite : $(A - \lambda I)x = 0$.
(Singular.)

$\det(A - \lambda I) = 0$ → Find λ first

↓

Find x (which is Nullspace)

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ → 6 $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 \cdots = 0$

→ $\begin{pmatrix} \lambda_1 = 4 \\ \lambda_2 = 2 \end{pmatrix}$ → find null space of $A - \lambda I$.

↓ 6.

Not so great ...

If $Ax = \lambda x$, B has eigenvalue α,

$Bx = \alpha x$

$(A + B)x = (\lambda + \alpha)x$? No!

$\because$ B's eigenvectors $\neq x$ (not always !!)

(example) $Q = 90°$ rotation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- trace $= 0 + 0$
- $\det Q = 1 = \lambda_1 \lambda_2$
- $\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

→ $\begin{pmatrix} \lambda_1 = +i \\ \lambda_2 = -i \end{pmatrix}$

(ex) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ → $\det(A - \lambda I) = (3 - \lambda)^2$

→ $\begin{cases} \lambda_1 = 3 \\ \lambda_2 = 3 \end{cases}$

$(A - \lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$
↑ singular.

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad x_2 =$ (No independent 2nd eigenvector)

~~for~~ degenerate case.

Diagonalizing a matrix $\boxed{S^{-1}AS=\Lambda}$

→ Suppose we have n indep. eig. vectors of A

put them in column of S

$AS = A[x_1\ x_2 \cdots x_n] = [\lambda x_1\ \lambda x_2 \cdots \lambda x_n]$

$= [x_1 \quad x_n] \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix}$

$= S\Lambda$ ← diagonal eigenvalue matrix Λ

$\Rightarrow S^{-1}AS = \Lambda$

$(A = S\Lambda S^{-1})$

If $Ax = \lambda x$

↳ $A^2x = \lambda Ax = \lambda^2 x$

↳ $A^k = S\lambda^k S^{-1}$

[theorem] $A^k \to 0$ as $k \to \infty$

if all $|\lambda_i| < 1$.

A is sure to have n indep eigenvectors (and be diagonalizable)

if all the λ's are different.

(no repeated λ's)

eig(rand(10, 10))

Repeated eigenvalues//

may or may not have n indep eigenvect

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \to \begin{pmatrix} \lambda_1 = 2 \\ \lambda_2 = 2 \end{pmatrix}$

→ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$'s nullspace..

↓

1D!

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

equation $u_{k+1} = Au_k$

→ start with given vector u_0

$u_1 = Au_0,\ u_2 = Au_1 \cdots = u_k = A^k u_0$

↳ to really solve: write

$u_0 = c_1x_1 + c_2x_2 + \cdots c_nx_n = Sc$

$A^{100}u_0 = c_1\lambda^{100}x_1 + c_2\lambda_2^{100}x_2 + c_n\lambda_n^{100}x_n$

$= \Lambda^{100}Sc$

- Fibonacci example; 0, 1, 1, 2, 3, 5, 8

$\boxed{F_{100}\,?}$

2nd order eq. ① $F_{k+2} = F_{k+1} + F_k$. $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

② $F_{k+1} = F_{k+1}$

$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\ \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \to \lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ (quadratic formula)

$\Rightarrow \begin{cases} \lambda_1 = 1.618.. \\ \lambda_2 = -0.618.. \end{cases} \Rightarrow F_{100} \approx c_1\left(\frac{1+\sqrt{5}}{2}\right)^{100}$

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $\rightarrow c_1x_1 + c_2x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

[① differential equation $\frac{du}{dt} = Au$
② exponential e^{At} of a matrix]

(ex) $\frac{du_1}{dt} = -u_1 + 2u_2$
$\frac{du_2}{dt} = u_1 - 2u_2$

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\hookrightarrow A = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \rightarrow \lambda = \begin{cases} 0 \\ -3 \end{cases}$

$\lambda_1 = 0$ $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $Ax_1 = 0x_1$
$\lambda_2 = -3$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Ax_2 = -3x_2$

solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

If $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\hookrightarrow c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

① stability $u(t) \rightarrow 0$?
need $e^{\lambda t} \rightarrow 0$. $Re(\lambda) < 0$.
$|e^{(-3+6i)t}| = e^{-3t}$
$|e^{6it}| = 1$

② steady state ? $\lambda_1 = 0$ and other $Re \lambda < 0$.

③ Blow up if any $Re \lambda > 0$.

Q. 2x2 stability $\begin{pmatrix} Re \lambda_1 < 0 \\ Re \lambda_2 < 0 \end{pmatrix}$?

① $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; trace $a + d < 0$

(trace < 0; but still blow up ?
$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$.)

② det > 0 $(\lambda_1 \lambda_2)$

$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$c = u(0)$

$\frac{du}{dt} = Au$ ②
Set $u = Sv$
$S \frac{dv}{dt} = ASv \rightarrow \frac{dv}{dt} = S^{-1}ASv$
$= \Lambda v$

$\therefore \frac{dv_1}{dt} = \lambda_1 v_1$

$\rightarrow v(t) = e^{\Lambda t} v(0)$

$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$

where $e^{At} = S e^{\Lambda t} S^{-1}$

- Matrix exponential $e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots \frac{(At)^n}{n!}$

$(I - At)^{-1} = I + At + \cdots (At)^n$ $\left(\because \frac{1}{1-x} = \sum x^n\right)$

$= SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2!} \cdots$

Markov matrices

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & .0 & .4 \end{bmatrix}$$

① All entries ≥ 0

② All columns add to 1.

key:
1. $\lambda = 1$ is an eigen values & $x_1 > 0$
2. All other $|\lambda_i| < 1$. (for steady state)

$$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 \cdots$$

($\lambda_1 = 1$, $\lambda_2^k \to 0$)

$\to c_1 x_1$ (steady state)

~~(∴ eigenvector $x_1 \geq 0$)~~

$$A - 1I = \begin{bmatrix} -.9 & 0.01 & .3 \\ .2 & -.01 & .3 \\ .7 & .0 & -.6 \end{bmatrix}$$

All columns add to zero → A−I is singular because rows are dependent

$(1,1,1)$ is $n((A-I)^T)$

then x_1 is in $n(A-I)$

. eigenvalues of A & eigenvalues of A^T are the same!

Application of Markov matrices

$u_{k+1} = A u_k$ (people moving to other city)

$$\begin{bmatrix} U_{california} \\ U_{massachusetts} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} U_{cal} \\ U_{mas} \end{bmatrix}$$

$$\begin{bmatrix} U_{cal} \\ U_{mas} \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

$$\to \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{pmatrix} \lambda_1 = 1 \\ \lambda_2 = .7 \end{pmatrix} \to \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\to x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \; \& \; x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 0.7^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$c_1 = \frac{1000}{3}$, $c_2 = \frac{2000}{3}$

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* projections with orthonormal basis

Any $v = x_1 q_1 + x_2 q_2 + \cdots x_n q_n$

$q_1^T v = x_1 q_1^T q_1 + 0 + \cdots 0 = x_1$ ($q_1^T q_1 = 1$)

$$v = \begin{bmatrix} q_1 \cdots q_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Qx$$

$x = Q^{-1} v = Q^T v$

$x_1 = q_1^T v$

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* Fourier series.

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x \ldots$

vectors

$v^T w = v_1 w_1 + \cdots v_n w_n$.

continuum vector → $f^T g = \int f(x) g(x) dx$.