⟨Linear Algebra, W. G. Strang⟩

- 29 Lectures. 4 Quiz reviews.

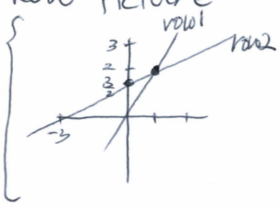
$\boxed{\text{Lec 1}}$ The geometry of linear equations.

- n-linear equation, n-unknowns.

$$2x - y = 0$$
$$-x + 2y = 3$$

$$\boxed{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}} \longleftrightarrow Ax = b$$

- Row picture



- column picture

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$; linear combination of columns.

Q. "Can I solve $Ax = b$ for every b?"

≡ "Do the linear combination of the columns in A fill n-dimensional space?"

$\boxed{\text{Lec 2}}$ Elimination with matrices

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \end{aligned} \longleftrightarrow Ax = b$$

$$\begin{matrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ & 4 & 1 \end{matrix} \xrightarrow{(2,1)} \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{matrix} \xrightarrow{(3,2)} \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{matrix}$$

1st Pivot

; Gaussian elimination ⟶ backsubstitution to find (x, y, z)

Elimination Matrices?

cf)
$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{matrix} 3\times\text{col1} \\ +4\times\text{col2} \\ +5\times\text{col3} \end{matrix}$$ $\boxed{\text{work by col}}$

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = \begin{matrix} 1\times\text{row1} \\ +2\times\text{row2} \\ +7\times\text{row3} \end{matrix}$$ $\boxed{\text{work by row}}$

→ Subtract 3×1st row from row 2?

(step 1)
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

(step 2)
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}_{U}$$

$$\Longleftrightarrow E_{32}(E_{21}A) = U.$$
$$= (E_{32}E_{21})A$$ ⟶ associative rule.

- permutation: Exchange rows 1 & 2?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \text{(row permutation)}$$

row operation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \quad \text{(column permutation)}$$

col operation

왼쪽 matrix; row operation
오른쪽 matrix; col operation.

- Inverses:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1을 세 배 더하기.　　row1을 세 배 곱해서 빼기

$$E^{-1} \cdot E = I$$

$\boxed{\text{Lec3}}$ Multiplication and inverse matrices.

method ① (Inner product)

$$\underset{A}{\begin{bmatrix} \text{row } i \\ \\ \end{bmatrix}}\underset{B}{\begin{bmatrix} \text{column } j \end{bmatrix}} = \underset{C}{\begin{bmatrix} C_{ij} \end{bmatrix}}$$

- $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

method ② (column)

col operation

$A(m \times n)$　$B(n \times p)$　$C(m \times p)$

- Columns of C are combination of columns of A

method ③ (Row)

row operation

$A(m \times n)$　$B(n \times p)$　$C(m \times p)$

- Rows of C are combination of Rows of B

method ④ (outer product)

$$\underset{(m\times 1)}{(\text{column of } A)} \times \underset{(1\times p)}{(\text{row of } B)}$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

AB = Sum of (Cols of A) × (Rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix}$$

method ⑤

Block multiplication.

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}\underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \underset{C}{\left[\begin{array}{c|c} & \\ \hline & \end{array}\right]}$$

$C_1 = A_1B_1 + A_2B_3$

* Inverses (square matrices)

$$A^{-1}A = I = AA^{-1}$$

$\boxed{\text{If this exist}} = \boxed{\text{invertible}} \equiv \boxed{\text{nonsingular}}$

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; singular.

↳ why singular?

Answer = "you can find a vector x with $Ax = 0$"

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$; invertible.

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}\underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ A × column j of A^{-1} = column j of I

→ 2 Gaussian elimination!

→ Gauss-Jordan (2 equations at once)

Gaussian-Jordan (Solve 2eqn at once)

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cc|cc}1&3&1&0\\2&7&0&1\end{array}\right] \longrightarrow \left[\begin{array}{cc|cc}1&3&1&0\\0&1&-2&1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1&0&7&-3\\0&1&-2&1\end{array}\right]$$

elimination하면 Identity가 Inverse matrix로 변함

$$\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{I}\quad \underbrace{\begin{bmatrix}7&-3\\-2&1\end{bmatrix}}_{A^{-1}}$$

$(\because EA = I \rightarrow EI = E)$

Lec 4. Factorization into A=LU

$(AB)(B^{-1}A^{-1}) = I$. if A & B invertible.

$(A^{-1})^T(A)^T = I^T = I$

{2x2 case}

$$\underset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}} \longrightarrow \underset{E_{21}}{\begin{bmatrix}1&0\\-4&1\end{bmatrix}}\underset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}} = \underset{u}{\begin{bmatrix}2&1\\0&3\end{bmatrix}}$$

inverse

(lower triangular) (upper triangular)

$$\underset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}} = \underset{L}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\underset{u}{\begin{bmatrix}2&1\\0&3\end{bmatrix}}$$

$$\underset{L}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\underset{D}{\begin{bmatrix}2&0\\0&3\end{bmatrix}}\underset{U}{\begin{bmatrix}1&1/2\\0&1\end{bmatrix}}$$

(diagonal)

(3×3 matrix case)

~~$E_{21}A$~~ $E_{32}E_{31}E_{21}A = U$ (no row exchanges)

$\rightarrow A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$

$= LU$

Suppose that …

$$\underset{E_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\quad \underset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}} \cdots$$

$$\overset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}}\overset{E_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}} = \begin{bmatrix}1&0&0\\\boxed{-2}&1&0\\10&\boxed{-5}&1\end{bmatrix} = E \text{ (left of A)}$$

neg multipliers

inverse (in reverse order)

$$\overset{E_{21}^{-1}}{\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}}\overset{E_{32}^{-1}}{\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}} = \begin{bmatrix}1&0&0\\\boxed{2}&1&0\\0&\boxed{5}&1\end{bmatrix} = L \text{ (left of } u)$$

multipliers.

$EA = u$

$\rightarrow A = LU$. is good (better than $EA=u$)

If no row exchanges, multipliers go directly into L.

Q. How many operations on n×n matrix A?

if n=100

1st pivot.

about ~100^2

operations = (multiply + subtract)

about ~99^2

⇓

$O(n)$?
$O(n^2)$?
$O(n!)$?

$\therefore$ operations count $\approx n^2+(n-1)^2+\cdots+1^2 \approx \boxed{\frac{1}{3}n^3}$ oA.

$(A|b) \rightarrow$ elimination cost for b?

$\sim n^2$.

$EA = U$, $Eb = b'$

$O(\frac{1}{3}n^3)$ $O(n^2)$

Q. If row exchange is allowed....

Permutations

(3×3)

$P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_{right} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$P_{left} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

6 possible permutations $(=3!)$

$P^{-1} = P^T$

If (4×4) permutations $\Rightarrow$ 24 P's $(\because 4!)$

Lec 5 Transposes, Permutations, Spaces R^n.

Permutations P : execute row exchanges.

what happen when $A = LU$?

$= \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & & \\ 0 & & \\ & & \end{bmatrix}$

P, here, is identity.

matlab needs permutation (row exchange) for accuracy...

($\because$ matlab hates very small pivot!!)

작은 피벗 & 큰 멀티플라이어 → 에러 커짐!

$\Rightarrow$ $PA = LU$

(invertible)

P = "identity matrix with reordered rows"

$\rightarrow$ How many? $\Rightarrow$ $n!$

& $P^{-1} = P^T \Leftrightarrow P^TP = I$.

Transpose

$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$

$(A^T)_{ij} = A_{ji}$

Symmetric matrices

$A^T = A$ example $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

✱ 임의의 R 에 transpose (R^T) & multiply

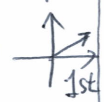
$\Rightarrow RR^T$ = symmetric!

$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & & \\ 7 & & \ddots \end{bmatrix}$

Vector Spaces.

(examples) R^2 = vectors $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}$

all 2-dimensional real

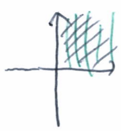


= "x − y plane"

R^3 = all 3-dimensional real vectors.

R^n = all ~~vs~~ real vectors with n compon-ents.

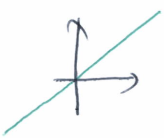
* Not a vector space?



← multiply by negative scaler

⇒ out of the set ⇒ Not a vector space!

* a vector space in R^2 ⇒ subspace.



= lines in R^2 through zero vector

* subspace of R^2 = ① all of R^2

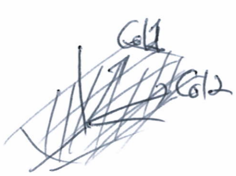
② any line of R^2 through zero

③ zero vector alone

∘ subspace of R^3?

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ columns in R^3

↳ all their combinations form a subspace.



Called column space $C(A)$

⟨Vector space requirement:⟩

- $v+w$ and cv are in the space
- all combs $\boxed{cv+dw}$ are in the space

Lec 6: Column space & null space

LA⑤

2 (two) subspaces: P and L

$P \cup L$ = all vectors in P or L or both.

union

This (is) (is not) a subspace!

$P \cap L$ = all vector in both P & L.

This (is) a subspace!

General Q: Subspaces S and T,

intersection $S \cap T$ is a subspace.

* Column space of A is subspace of R^4.

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ ↳ all linear comb. of columns

Q1 (↳ We get smaller subspace than R^4. then, "How small"?)

Q2 "Does $Ax=b$ have a solution for every b?" ↳ If not, "which rightside (b) are OK?"

$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

" || which b's allow the system to be solved?

* Null space of A, $N(A)$

= "All solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax=0$"

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \rightarrow c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

∘ check that solutions to $Ax=0$ always give a subspace

Lec 7. Solving $Ax=0$; pivot variable, special solution

Q. What's the algorithm that solves $Ax=0$?

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$$

elimination

$$\rightarrow\begin{bmatrix}1&2&2&2\\0&\boxed{0}&2&4\\0&0&2&4\end{bmatrix}$$

zero means... "2nd column is dependent on its previous column"

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}$$

"echelon" = stair-case

free columns

↳ the number of pivots (2) = rank of A

free of choice of numbers..

$$x=\begin{bmatrix}-2\\\boxed{1}\\0\\\boxed{0}\end{bmatrix}$$

$x_1+2x_2+2x_3+2x_4=0$

$2x_3+4x_4=0$

맘대로 정할수없어.

그러면 이거 되는 정답존재

free variable of choice.

1로 정해버려 0으로 정해버려.

$$x=c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

$\Rightarrow$ { $m\times n$ matrix, n variables w/ r rank

$\Rightarrow$ $n-r=4-2=2$ free variables }

- Elimination → pivot # = rank
  → free variable # = nullity.

R = reduced row echelon form. (rref) has... zeros above & below pivots = 1

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}\boxed{1}&2&0&-2\\0&0&\boxed{2}&4\\0&0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}\boxed{1}&2&0&-2\\0&0&\boxed{1}&2\\0&0&0&0\end{bmatrix}=R$$

Reconstruction of R:

$$\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{\text{pivot cols}}\quad\underbrace{\begin{bmatrix}2&-2\\0&2\end{bmatrix}}_{\text{free cols}}$$

$I \quad F$

$$\therefore R=\begin{bmatrix}I&F\\0&0\end{bmatrix}$$

→ r pivot rows

r pivot cols

$(n-r)$ free cols

$Rx=0$?

$\Rightarrow$ null space matrix (columns = special solution)

$RN=0$

where $N=\begin{bmatrix}-F\\I\end{bmatrix}$

then $\begin{bmatrix}I&F\end{bmatrix}\begin{bmatrix}x_{pivot}\\x_{free}\end{bmatrix}=0$

↳ $x_{pivot}=-Fx_{free}$

앞문제 transpose

$$A^T=\begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix}\rightarrow\begin{bmatrix}\boxed{1}&2&3\\0&0&0\\0&2&2\\0&4&4\end{bmatrix}\rightarrow\begin{bmatrix}\boxed{1}&2&3\\0&\boxed{2}&2\\0&0&0\\0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}\boxed{1}&0&1\\0&\boxed{1}&1\\0&0&0\\0&0&0\end{bmatrix}$$

$I \quad F$

pivot cols (2) free cols (3−2=1)

$r=2$ again

↳ $x=c\begin{bmatrix}-1\\-1\\\boxed{1}\end{bmatrix}=c\begin{bmatrix}-F\\I\end{bmatrix}$

free variable

Lec 8 Solving $Ax=b$; row reduced form R

$$\begin{cases} 1x_1+2x_2+2x_3+2x_4 = b_1 \\ 2x_1+4x_2+6x_3+8x_4 = b_2 \\ 3x_1+6x_2+8x_3+10x_4 = b_3 \end{cases}$$

pivot cols

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \to \left[\begin{array}{ccccc} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right]$$

Augmented matrix = $[A \; b]$

'0' 이어야만 함 ⇓ solvability equation.

$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \to$ ok.

- Solvability Condition on b

; { $Ax=b$ solvable when b is in $C(A)$

⟺ { If a combination of rows of A gives "zero row", then same comb. of the entries of b must be "zero"

- To find complete sol'n to $Ax=b$.

① $X_{particular}$: Set all free variables to zero.

Solve $Ax=b$ for pivot variables

$(x_2=0, x_4=0)$

↳ $x_1+2x_3 = 1 \qquad x_1=-2$

$2x_3 = 3 \to x_3 = 3/2$

$\therefore x_p = \begin{bmatrix} -2 \\ 0 \\ 3/4 \\ 0 \end{bmatrix}$

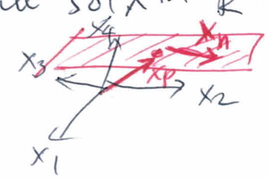
② $X_{nullspace}$

③ $X_{complete} = X_{particular} + X_{nullspace}$

$= X_p + X_n$

$\therefore X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 2 \end{bmatrix} + c\begin{bmatrix} -2 \\ \boxed{1} \\ 0 \\ \boxed{0} \end{bmatrix} + d\begin{bmatrix} 2 \\ \boxed{0} \\ -2 \\ \boxed{1} \end{bmatrix}$

- Plot all sol X in R^4



- $m \times n$ matrix A of rank r ($r \le m$, $r \le n$)

- Full column rank means $r=n$? (col number)

↳ No free variables ≡ "$N(A) = \{$zero vector$\}$"

↳ Solution to $Ax=b$: $X = X_p$ { unique solution if it exists }

- Full row rank means $r=m$?

↳ Can solve $Ax=b$ for every b (Exist)

Left with $n-r$ free variables.

정리 { $r=n$ (cols) ⇒ 존재하지 않을 수 있지만, 존재하면, unique solution!

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$ (∵ remaining row = solvability condition)

$r=m$ (rows) ⇒ solution 항상 존재하지만 unique solution은 아님!

$R = [\,I \;\; F\,]$ (∵ remaining cols = free columns ↳ free var.)

- Full rank means $\boxed{r=m=n}$

⟺ Invertible (solution exists & unique)

$R = [\,I\,]$

generally

- $r<m, r<n \to R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

Lec 9 Independence, basis, dimension

- Suppose A is m by n with $m<n$

→ "Then there are nonzero solutions to $Ax=0$"
(more unknowns than equations)

↳ ∵ There will be at least more than one free variables!!

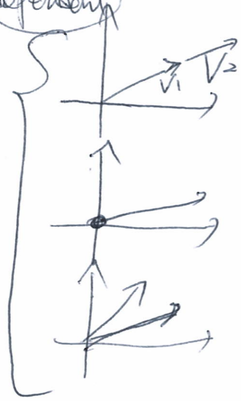
- **Independence**

Vectors $X_1, X_2, \cdots, X_n$ are independent if no combination gives zero vector (except the zero comb.)

$$c_1X_1 + c_2X_2 + \cdots c_nX_n \neq 0 \text{ for } \forall c_i$$

except all zero

dependent



$2V_1 - V_2 = 0$

$0V_1 + 6V_2 = 0$

Repeat when $V_1, \cdots V_n$ are columns of A

- They are <u>independent</u> if nullspace of A is {zero vector}
[rank = n]
- they are <u>dependent</u> if $Ac = 0$ for some nonzero c vector.
[rank < n]

Vector $V_1 \sim V_\ell$ span a space means: The space consists of all combs. of those vectors

Basis for a space is a sequence of vectors $V_1, V_2, \cdots, V_d$ with 2 properties

① they are independent
② they span the space

n vectors give basis if $n \times n$ matrix with those cols is invertible.

= Every basis for the space has the same number of vectors

= Def. Dimension of the space!

* $\dim C(A) = r$

* $\dim N(A) = \#$ of free variables $= (n-r)$

Lecture 10 The four fundamental subspaces

$$A=\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \rightarrow \begin{bmatrix}1&2&3&1\\0&1&1&0\\0&0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}=R$$

I F ϕ

$C(R) \neq C(A)$

⊛ row operation preserves row space.

But it changes column space

- Basis for row space is first r rows of R

$$A=\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \leftarrow \begin{bmatrix}1&2&3&1\\0&1&1&0\\0&0&0&0\end{bmatrix} \leftarrow \begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}=R$$

- 4th space : $N(A^T)$

$$A^T y = 0 \xrightarrow{\text{transpose}} y^T A^{T^T} = 0^T$$

$[\quad][\,] = [0]$ $[\;y^T\;][\;A\;] = [\;0\;]$

- How do we get basis for left nullspace?

rref$[A_{m\times n}\ I_{m\times m}] \rightarrow [R_{m\times n}\ E_{m\times m}]$

$\Rightarrow E_{m\times m}A_{m\times n} = R_{m\times n}$.

{In chap 2, R was I $\therefore E=A^{-1}$}
(when A is invertible)

$$\begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \rightarrow \begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}$$

E basis for left null space.

[New vector space !] (M) LA9

↳ All 3x3 matrices !! ($A+B$, cA)

↳ subspaces of M ; all upper triangular matrices U

all symmetric matrices S

all diagonal matrices D

dim of this subspace is 3.

$$\begin{bmatrix}1&&\\&0&\\&&0\end{bmatrix}\begin{bmatrix}0&&\\&1&\\&&0\end{bmatrix}\begin{bmatrix}0&&\\&0&\\&&1\end{bmatrix}$$

[Lecture 11. Basis of new vectorspaces
& Rank one matrices
& Small world graphs]

$M \rightarrow$ all 3x3 matrices.

Basis for $M \Rightarrow \begin{bmatrix}1&&\\&&\\&&\end{bmatrix}\begin{bmatrix}&1&\\&&\\&&\end{bmatrix}\begin{bmatrix}&&1\\&&\\&&\end{bmatrix}\cdots$ 9 dim

$\dim(S)=6$, $\dim(U)=6$, $\dim(D)=3$.

- $S \cap U$ (intersection) = symm & upper triangular
$= D =$ diagonal 3x3's $\dim(S\cap U) = D$.

- $S \cup U \Rightarrow S + U$ (sum) = any element of S + any element of U
filling subspace
$=$ all 3x3's

$\dim(S+U) = 9$.

$6 + 6 = 3 + 9$

(ex) $\ddot{y} + y = 0$

$y = \cos x, \sin x, e^{ix} \rightarrow$ complete solution $y = c_1\cos x + c_2\sin x$

$\dim(\text{solution space}) = 2$.

• What is rank ?

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

($\dim C(A) = \text{rank} = \dim C(A^T)$)

$r = 1.$

$$\Rightarrow \underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}$$

$\boxed{\text{Rank 1 matrix} \ \hookrightarrow A = uv^T}$

⇓

중요해!

why?

Rank N matrix = N개의 Rank 1 matrix 로 쪼갤수 있다.

ex) M = all 5×17 matrices.

• subset of rank 1 matrices.

↳ not a subspace.

ex) In $\mathbb{R}^4$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ S = all v in $\mathbb{R}^4$ with $v_1+v_2+v_3+v_4 = 0$.

S = null space of A

$(A = [1 \ 1 \ 1 \ 1])$

rank = 1 = r

$\dim N(A) = n - r = 4 - 1 = 3$

basis for S ←

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

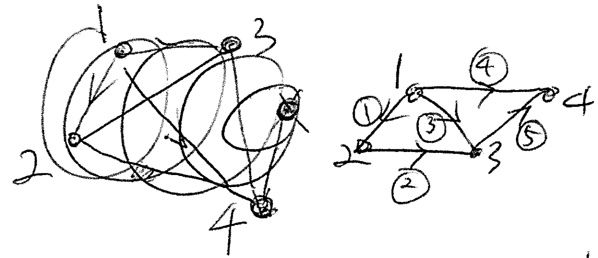
$C(A) = \mathbb{R}^1$, $N(A^T) = \{0\}$

column space

$$\begin{cases} 3+1 = n = 4 \\ 1+0 = 1 = m. \end{cases}$$

< Application of Linear Algebra
⇒ Applied math. >

Graph = { nodes, edges }



n = 4 nodes. m = 5 edges.

• Incidence matrix

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} \text{node1} & \text{node2} & \text{node3} & \text{node4} \end{array}$$

$$A = \left[\begin{array}{cccc} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \text{edge1} \\ \text{edge2} \\ \text{edge3} \\ \text{edge4} \\ \text{edge5} \end{array}$$

loop

rows are dependent

(출발 : −1, 도착 : +1)

$$Ax = 0 = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix}$$

$x = x_1, x_2, x_3, x_4$

↳ potential at nodes.

$\boxed{y = Ce}$ (C) (A) $\boxed{e = Ax}$

$x_2 - x_1$, etc.

Ohm's law

Current $y_1 y_2 \cdots y_5$ on edges

Potential differences → zero!

$x = c\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ → null space.

⇓

$\begin{cases} \dim N(A) = 1. \\ \text{rank} = 3. \end{cases}$

$A^T y = 0$ ← If current source exist

Kirchhoff's current law. $A^T y = f$

∴ $\boxed{A^T C A x = f}$

↳ symmetric

$A^T y = 0$, $\dim N(A^T) = m - r = 5 - 3 = 2$

$n \times m$

4×5

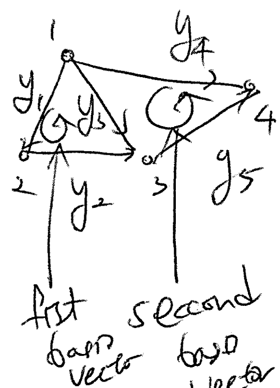
$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y_1 - y_3 - y_4 = 0.$

$+y_1 - y_2 = 0$

$y_2 + y_3 - y_5 = 0$

$y_4 + y_5 = 0.$



Basis for $N(A^T)$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- graph w/o any loops $\equiv$ Tree

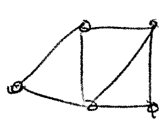
$\dim N(A^T) = m - r$

$\boxed{\# \text{loops} = \# \text{edges} - (\# \text{nodes} - 1)}$

(rank $= n-1$)

$\boxed{\# \text{node} - \# \text{edges} + \# \text{loops} = 1}$

Euler's formula.



$5 - 7 + 3 = 1.$